RECap: Bloch Sphere

Pure & Mixed State

2-particle : Quantum Entanglement

- Pure State : Rank
- Mixed State : $S(\rho_R)$; $\rho = \sum p_i |i\rangle\langle i|$ not unique

$$\min \sum p_i \, E(|\psi\rangle)$$

Peres-Horodecki ; Negativity , NPT (Entangled)

PPT (Entangled)

Bell-Inequalities

Cluster State.

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$(|0\rangle + |1\rangle)(|0\rangle + |1\rangle)$$

$$\xrightarrow{CZ} (\underbrace{|00\rangle + |01\rangle} + |10\rangle - |11\rangle)$$

$$= \underbrace{|0\rangle|+\rangle + |1\rangle|-\rangle}_{\text{maximally-entangled state}}$$

CZ

$$|00\rangle \to |00\rangle$$
$$|01\rangle \to |01\rangle$$
$$|10\rangle \to |10\rangle$$
$$|11\rangle \to e^{i\pi}|11\rangle$$

$$\underbrace{|+\rangle|0\rangle + |-\rangle|1\rangle}$$

$$(|\phi\rangle|\psi\rangle \pm |\phi^{\perp}\rangle|\psi^{\perp}\rangle)$$

$$|+\rangle|+\rangle \xrightarrow{CZ} |0\rangle|+\rangle + |1\rangle|-\rangle$$
$$= |+\rangle|0\rangle + |-\rangle|1\rangle$$

$$\underbrace{(|0\rangle + |1\rangle)(|0\rangle + |1\rangle)}(|0\rangle + |1\rangle)$$

$$\xrightarrow{CZ} (|0\rangle|+\rangle + |1\rangle|-\rangle)(|0\rangle + |1\rangle)$$

$$= |0\rangle \underbrace{|+\rangle(|0\rangle + |1\rangle)} + |1\rangle \underbrace{\overset{|0\rangle - |1\rangle}{|-\rangle}(|0\rangle + |1\rangle)}$$

$$\longrightarrow |0\rangle|+\rangle|0\rangle + |0\rangle|-\rangle|1\rangle + |1\rangle\{|00\rangle + |01\rangle - |10\rangle + |11\rangle\}$$

$$= |0\rangle|+\rangle|0\rangle + |0\rangle|-\rangle|1\rangle + |1\rangle|-\rangle|0\rangle + |1\rangle|+\rangle|1\rangle$$

Exercise: $|\phi_1\rangle|\phi_2\rangle|\phi_3\rangle + |\phi_1^\perp\rangle|\phi_2^\perp\rangle|\phi_3^\perp\rangle$

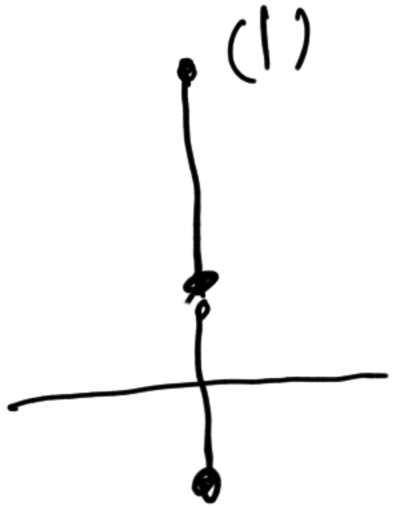
GHZ

$N=4$

$|\psi^1_{cs}\rangle$

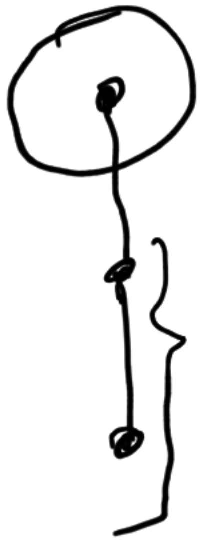
$|\psi^2_{cs}\rangle$

$$|\psi^2_{cs}\rangle \neq U_1 \otimes U_2 \otimes U_3 \, |\psi^{(1)}_{cs}\rangle$$



$$\simeq \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)$$

$$\frac{1}{\sqrt{2}}\left(|HHH\rangle + |VVV\rangle\right)$$



concurrence

$$|V\rangle = \cos\theta|c\rangle + \sin\theta|nc\rangle$$

$$|H\rangle = \sin\theta|c\rangle - \cos\theta|nc\rangle$$

$$\langle V|H\rangle = 0$$

$$\frac{1}{\sqrt{2}}\left\{\left(\sin\theta|c\rangle - \cos\theta|nc\rangle\right)|HH\rangle + \left(\cos\theta|c\rangle + \sin\theta|nc\rangle\right)|VV\rangle\right\}$$

$$= \frac{1}{\sqrt{2}}|c\rangle\left(\sin\theta|HH\rangle + \cos\theta|VV\rangle\right) \text{ or } |nc\rangle\frac{1}{\sqrt{2}}\left(\cos\theta|HH\rangle + \sin\theta|VV\rangle\right)$$

$$|\psi^{x}\rangle = \underbrace{\sin\theta\,|HH\rangle + \cos\theta\,|VV\rangle}_{2\sin\theta\cos\theta} \qquad \text{prob} = \frac{1}{2}$$

$$|\psi'\rangle = \underbrace{-\cos\theta\,|HH\rangle + \sin\theta\,|VV\rangle}_{|-2\sin\theta\cos\theta|} \qquad \text{prob} = \frac{1}{2}$$

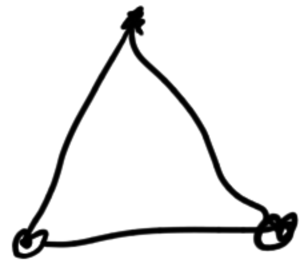
$$\text{Average Entanglement} = 2\sin\theta\cos\theta$$

$$= \sin 2\theta$$

$$\underset{\max}{=} 1$$

Localizable Entanglement

$|\psi\rangle$



$|\psi'\rangle$

Briegel & Raussendorf PRL (2002)

$$|\psi\rangle = U_1 \otimes U_2 \otimes U_3 |\psi'\rangle$$

Exercise: $|\phi\rangle = \frac{1}{\sqrt{3}}\left(|001\rangle + |010\rangle + |100\rangle\right)$

Compute localizable entanglement

$$\frac{1}{\sqrt{3}}\Big\{|0\rangle\big(|01\rangle + |10\rangle\big) + |1\rangle\big(|00\rangle\big)\Big\}$$

$$\big(\cos\theta\,|c\rangle + \sin\theta\,|nc\rangle\big) \qquad \big(\sin\theta\,|c\rangle - \cos\theta\,|nc\rangle\big)$$

$|c\rangle$ $\qquad\qquad$ $|nc\rangle$

$$\cos\theta\big(|01\rangle + |10\rangle\big) + \sin\theta\,|00\rangle \qquad \sin\theta\big(|01\rangle + |10\rangle\big) - \cos\theta\,|00\rangle$$

$$\frac{1}{3}\big(1+\cos^2\theta\big) \qquad \frac{1}{3}\big(1+\sin^2\theta\big)$$

$$E_1 = 2\cos^2\theta \qquad E_2 = 2\sin^2\theta$$

$$E_{av} = \frac{1}{3}(1+\cos^2\theta)\,2\cos^2\theta \quad + \quad \frac{1}{3}(1+\sin^2\theta)\,2\sin^2\theta$$

$$= \frac{2}{3} + \frac{2}{3}\left(\underbrace{\cos^4\theta + \sin^4\theta}_{\leq \left(\frac{1}{\sqrt{2}}\right)^4}\right) \quad \sim$$

$$\frac{\partial E}{\partial \theta} = \frac{2}{3}\left(-4\cos^3\theta\sin\theta + 4\sin^3\theta\cos\theta\right) = 0$$

$$\sin^3\theta\cos\theta = \cos\theta\sin\theta$$

$$\tan^3\theta = \tan\theta \qquad \theta = \frac{\pi}{4}$$

$$\underset{\theta}{\mathrm{opt}}\; E_{av} = \frac{2}{3} + \frac{1}{6} \sim \frac{5}{6}$$

$X|0\rangle = |1\rangle \qquad Z|+\rangle = |-\rangle$

$X|1\rangle = |0\rangle \qquad Z|-\rangle = |+\rangle$

$$|\psi\rangle = \underbrace{|0\rangle_1 |+\rangle_2 |0\rangle_3} + \underbrace{|0\rangle|-\rangle|1\rangle} + \underbrace{|1\rangle|-\rangle|0\rangle} + \underbrace{|1\rangle|+\rangle|1\rangle}$$

$$\underbrace{X_1 Z_2}|\psi\rangle = |1\rangle|-\rangle|0\rangle + |1\rangle|+\rangle|1\rangle + |0\rangle|+\rangle|0\rangle + |0\rangle|-\rangle|1\rangle$$

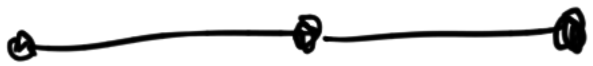
$$= |\psi\rangle$$

$$\underbrace{Z_1 X_2 Z_2}|\psi\rangle = |\psi\rangle$$

$$\underbrace{Z_2 X_3}|\psi\rangle = |\psi\rangle$$

$$\{X_1 Z_2,\ Z_1 X_2 Z_2,\ Z_3 X_3\}$$

Stabilizers



$\{X_1 Z_2 \,,\; Z_1 X_2 Z_2 \,,\; Z_2 X_3\}$

bit flip in qubit 1

phase flip error in qubit 2

$\longrightarrow X_1 Z_2 \underbrace{Z_1 X_2 Z_2 |\psi\rangle} = |\tilde{\psi}\rangle$

$\underbrace{X_1 Z_1} \underbrace{Z_2 X_2}_{-iY_2} Z_2$

① Arbitrary unitary operation on single qubit

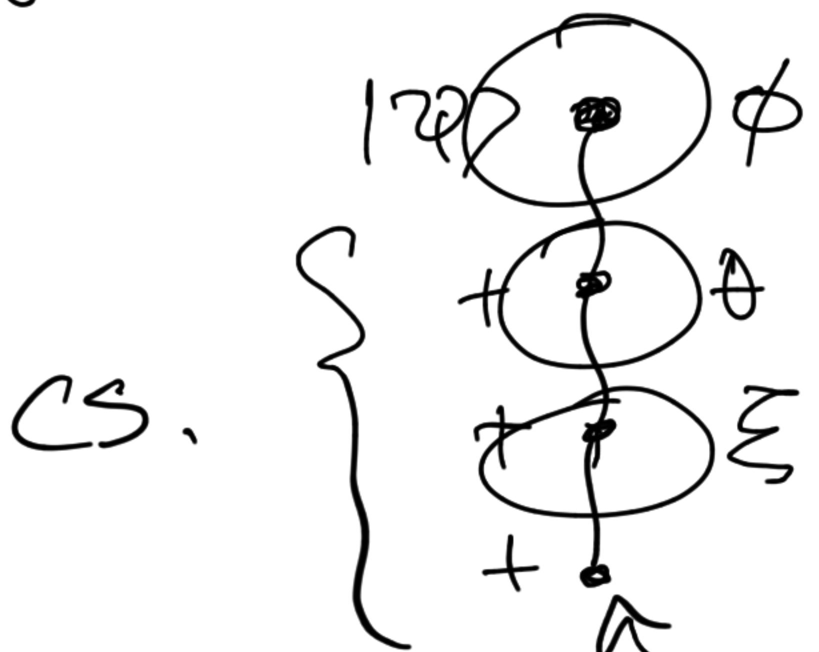
② Control-NOT gate

$\Downarrow$

Universal computation

R. Jozsa : Measurement-based Q.C.

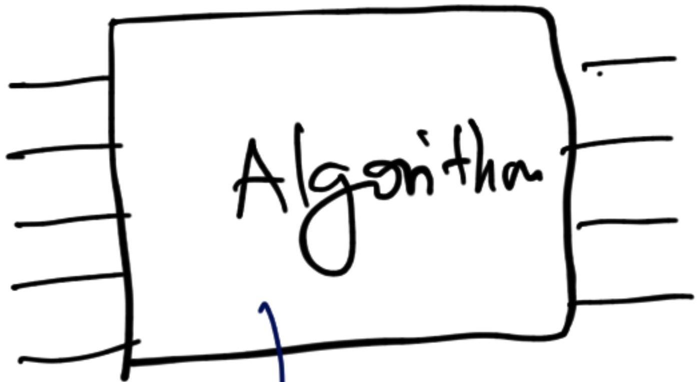
$U(\phi, \theta, \xi)$



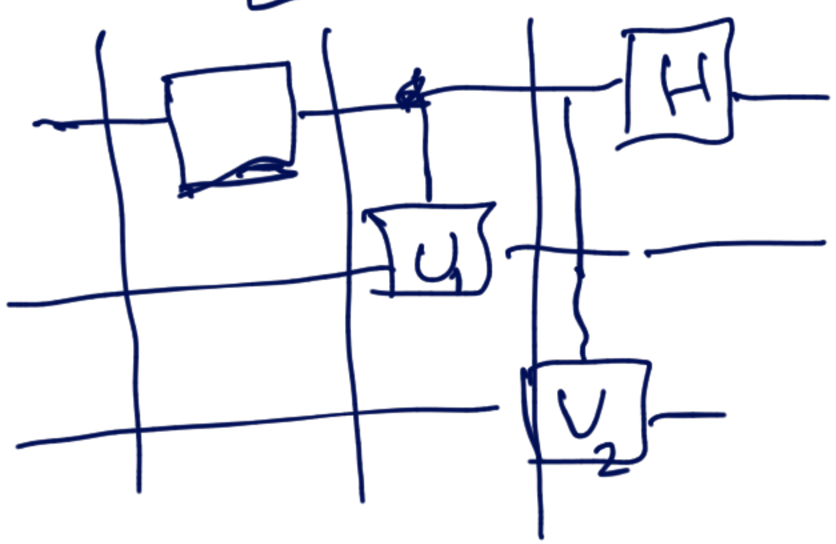
$U(\phi, \theta, \xi)|\psi\rangle$

Unitary Paradigm

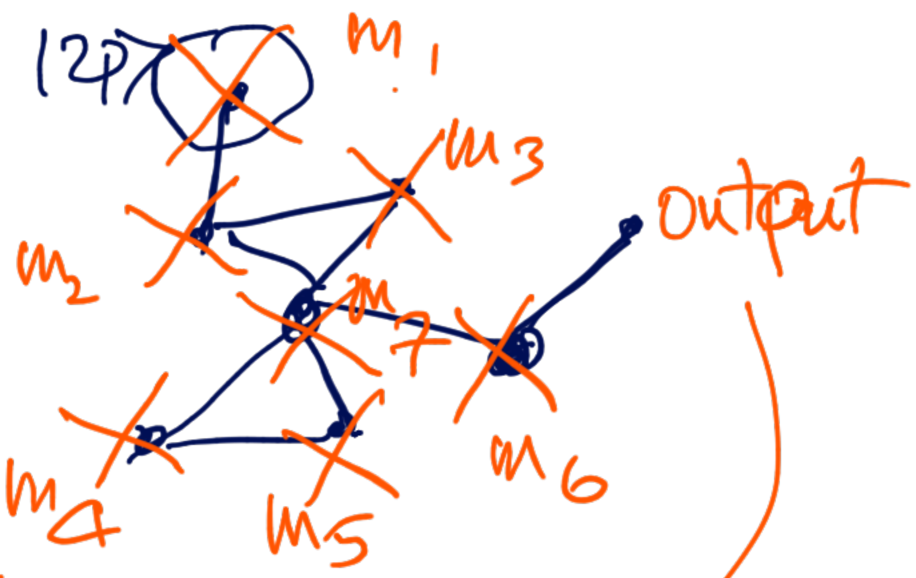
"sep" input



output $\Rightarrow$ Desired outcome



MBQC (one-way QC) Algorithm



Highly entangled

Desired $U(m_1, \ldots, m_7)$ output

$|CS\rangle$

$$\{X_1Z_2,\ Z_1X_2Z_3,\ Z_2X_3\}$$

$$H = -X_1Z_2 - Z_1X_2Z_3 - Z_2X_3$$

3-body Term

$$H|CS\rangle = \underbrace{-3}|CS\rangle$$

$$\rho(T) = \frac{e^{-\beta H}}{Z} \qquad \beta = \frac{1}{kT}$$

$$\longrightarrow |CS\rangle$$

Shown by Mike Nielsen that no 2-body Hamiltonian with nearest neighbor interaction can produce CS.

PRL 107, 060501 (2011)

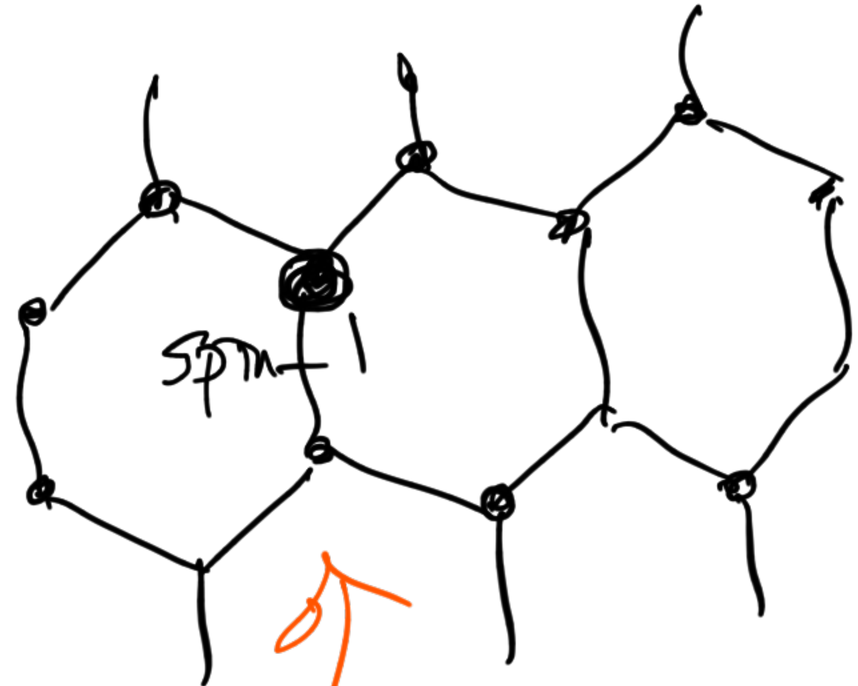
$$\rho = \frac{e^{-\beta H}}{Z}$$

cooled

$$\rho_{\text{gnd}} = |cs\rangle\langle cs|$$

PRL 113, 80501 (2014)

MBQC adiabatically
with 2-body interaction



H

Ground State

Project to "some" subspace
$\hookrightarrow |cs\rangle$

Applications of Q. Entanglement

(1) Q. Teleportation

$$\underbrace{|\psi\rangle_A |\Phi^+_{AB}\rangle} = \left(\alpha|0\rangle + \beta|1\rangle\right)_A \left(\frac{1}{\sqrt{2}}\left(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B\right)\right)$$

$$=$$

$$= \frac{1}{2}|\Phi^+_{AA}\rangle|\psi\rangle_B$$
$$+ \frac{1}{2}|\Phi^-_{AA}\rangle\sigma_z|\psi\rangle$$
$$+ \frac{1}{2}|\Psi^+\rangle\sigma_x|\psi\rangle$$
$$+ \frac{1}{2}|\Psi^-\rangle\sigma_y|\psi\rangle$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right)$$

Exercise!

(2) Q. Dense Coding

$$\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

$\mathbb{1}$: 00

σ_x 01

σ_y 11

σ_z 10

$1 \otimes \sigma_x$

$1 \otimes \sigma_y$

$1 \otimes \sigma_z$

$$\frac{1}{\sqrt{2}}\left(|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B\right)$$

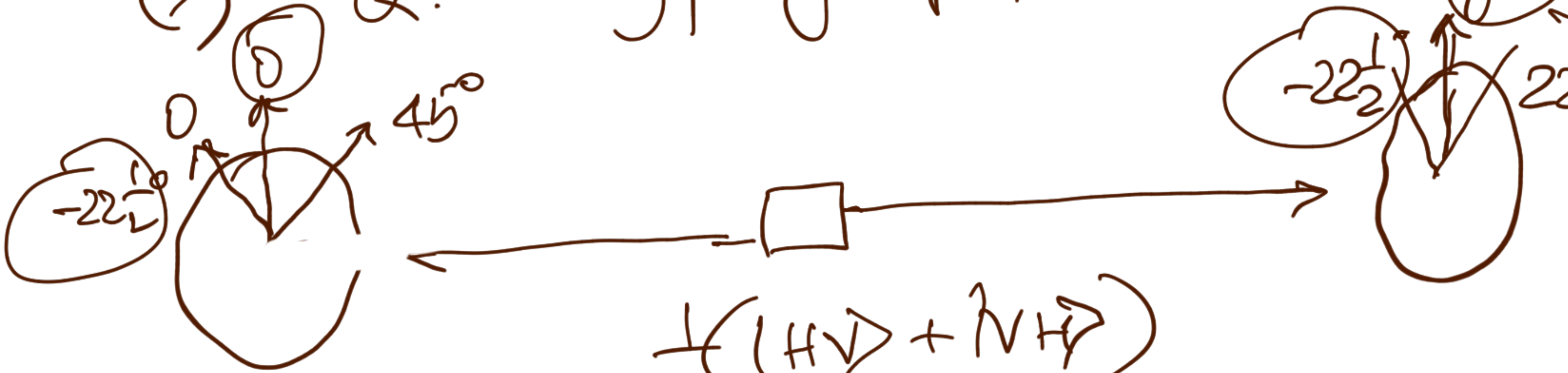
2 bits of classical info

$\mathbb{1}, \sigma_x, \sigma_y, \sigma_z$

1 qubit + 1 ebit $\rightarrow$ 2 cbit

(3) Q. Cryptography

BB84 Ekert91



$$\frac{1}{\sqrt{2}}\left(|HV\rangle + |VH\rangle\right)$$

Alice \ Bob	$-22\frac{1}{2}°$	0°	$22\frac{1}{2}°$
$-22\frac{1}{2}°$			
0°	✓		✓
45°	✓		✓

$\Longleftrightarrow$ BB84

Bell Inequality

$2\sqrt{2}$